Comparative Study of Circular Plate Displacement Based on MITC3 Finite Element Formulation

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***Abstract*— This paper presents a comparison of MITC3 plate element with different mesh directions. MITC3 is a three-node triangular element with three degrees of freedom proposed by Lee and Bathe. The circular plate problem with left and right mesh directions is used in the numerical test to verify the performance and show the convergence of three-node triangular elements.**

***Keywords*—Finite element method, MITC3 plate element, triangular element**

I. INTRODUCTION

The finite element method has been used in the past several decades to approximate the exact solution of a mathematical model. Two plate theories have been described. Kirchhoff–Love plate theory, which is used C^1 continuity, neglects shear deformation and can be used in thin plates; Reissner–Mindlin plate theory considers shear deformation and uses C^0 continuity [1]. Reissner–Mindlin theory is simpler and more efficient than Kirchhoff–Love plate theory and can be used to analyze thick and thin plates. However, this element encounters shear locking in thin plates, causing overstiffness. Several techniques were developed to alleviate shear locking. Assumed natural strains (ANS) was proposed by Hughes et al. in 1981 [6]. The Mixed Interpolation of Tensorial Component (MITC3) element is a three-node triangular element developed based on Reissner–Mindlin plate theory and independent shear strain field proposed by Lee and Bathe in 2004 [2], and this development is still being improved [8-10]. The key role of the MITC3 element is to interpolate displacement and strain independently and connect at the tying point. The interpolation of shear strain is based on the ANS [7-8].

The purpose of this paper is to compare the performance of the MITC3 element with different mesh directions. The numerical test of the circular bending plate with left and right mesh directions is used to compare and evaluate the performance.

MITC3 methods and finite element analysis in FEAP8.3 software are used in this study.

A. Geometry of Plate Element

The geometry of the three-node triangular plate elements is interpolated by

$$x = \sum_{i=1}^{3} N_i x_i \,, \tag{1}$$

$$y = \sum_{i=1}^{3} N_i y_i \,, \tag{2}$$

where the shape function of node-*i* is

$N_1 = 1 - \xi - \eta$, $N_2 = \xi$, $N_3 = \eta$

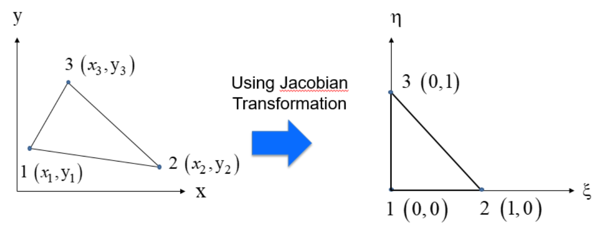

Fig. 1. Geometric aspect of triangular element

B. Displacement of Plate Element

MITC3 is a three-node element with three degrees of freedom [12]. They are one translation in the *z* direction (*w*) and two rotations in the *z-x* plane (β_x) and *z-y* plane (β_y). The displacement interpolation of the MITC3 plate element can be expressed as

$$w = \sum_{i=1}^{3} N_i w_i \,, \tag{3}$$

$$\beta_x = \sum_{i=1}^{3} N_i \beta_{xi} \,, \tag{4}$$

$$\beta_y = \sum_{i=1}^{3} N_i \beta_{yi} \,, \tag{5}$$

where,

β_{xi} : Rotation in *z-x* plane of node-*i*

β_{xi} : Rotation in *z-y* plane of node-*i*

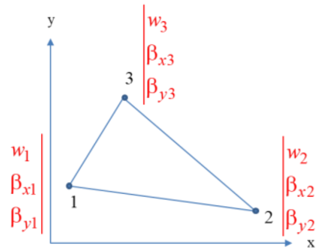

Fig. 2. Degrees of freedom of triangular element

C. *Bending Strain Matrix*

The relation of bending and curvature is given by

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ 2\varepsilon_{xy} \end{Bmatrix} = z \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix}. \tag{6}$$

The relation of curvature and rotation is given by

$$\begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} = \begin{Bmatrix} \beta_{x,x} \\ \beta_{y,y} \\ \beta_{x,y} + \beta_{y,x} \end{Bmatrix}. \tag{7}$$

The relation between curvature and nodal displacement is given by

$$\{\chi\} = [B_b]\{u_n\}. \tag{8}$$

The first derivation of rotation of Equations (4) and (5) can be expressed as

$$\beta_{x,x} = \sum_{i=1}^{3} N_{i,x}\,\beta_{xi} = N_{1,x}\,\beta_{x1} + N_{2,x}\,\beta_{x2} + N_{3,x}\,\beta_{x3},$$

$$\beta_{y,y} = \sum_{i=1}^{3} N_{i,y}\,\beta_{yi} = N_{1,y}\,\beta_{y1} + N_{2,y}\,\beta_{y2} + N_{3,y}\,\beta_{y3},$$

$$\beta_{x,y} + \beta_{y,x} = \sum_{i=1}^{3} N_{i,y}\,\beta_{xi} + \sum_{i=1}^{3} N_{i,x}\,\beta_{yi}. \tag{9}$$

The bending strain matrix for the MITC3 element can be expressed as:

$$[B_b] = \begin{bmatrix} 0 & N_{1,x} & 0 & 0 & N_{2,x} & 0 & 0 & N_{3,x} & 0 \\ 0 & 0 & N_{1,y} & 0 & 0 & N_{2,y} & 0 & 0 & N_{3,y} \\ 0 & N_{1,y} & N_{1,x} & 0 & N_{2,y} & N_{2,x} & 0 & N_{3,y} & N_{3,x} \end{bmatrix} \tag{10}$$

and

$$\langle u_n \rangle = \langle w_1 \quad \beta_{x1} \quad \beta_{y1} \quad w_2 \quad \beta_{x2} \quad \beta_{y2} \quad w_3 \quad \beta_{x3} \quad \beta_{y3} \rangle$$

D. *Shear Strain Matrix*

The shear deformation field is assumed linear in each element [11]:

$$\{\gamma\} = \begin{Bmatrix} \gamma_x \\ \gamma_y \end{Bmatrix} = \sum_{i=i}^{3} N_i \begin{Bmatrix} \gamma_{xi} \\ \gamma_{yi} \end{Bmatrix}. \tag{11}$$

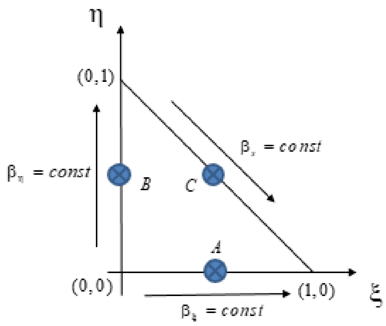

Fig. 3. Tying points of MITC3 element

The interpolation of the transverse shear strain field can be expressed as

$$\beta_\xi = a_1 + b_1\xi + c_1\eta, \tag{12}$$

$$\beta_\eta = a_2 + b_2\xi + c_2\eta, \tag{13}$$

$$\beta_s = \frac{\beta_\xi - \beta_\eta}{\sqrt{2}}, \tag{14}$$

where β_ξ is assumed constant along the ξ direction and β_η is assumed constant along the η direction

$$\beta_\xi = a_1 + c_1\eta, \tag{15}$$

$$\beta_\eta = a_2 + b_2\xi. \tag{16}$$

At nodes 1 and 2

$$\beta_\xi(0,0) = a_1 \;;\; \beta_\xi(1,0) = a_1 \;. \tag{17}$$

At nodes 1 and 3

$$\beta_\eta(0,0) = a_2 \;;\; \beta_\eta(0,1) = a_2 \;. \tag{18}$$

The constant shear deformation along the edge is as follows.

At node 3:

$$\beta_\xi(0,1) = a_1 + c_1 \tag{19}$$

At node 2:

$$\beta_\eta(1,0) = a_2 + b_2 \tag{20}$$

$$\beta_s(1,0) = \beta_s(0,1) \tag{21}$$

$$\frac{1}{\sqrt{2}}\left(\beta_\xi(1,0) - \beta_\eta(1,0)\right) = \frac{1}{\sqrt{2}}\left(\beta_\xi(0,1) - \beta_\eta(0,1)\right) \tag{22}$$

$$\frac{(a_1) - (a_2 + b_2)}{\sqrt{2}} = \frac{(a_1 + c_1) - (a_2)}{\sqrt{2}} \tag{23}$$

$$b_2 = -c_1 = c \quad (24)$$

Thus, β_ξ and β_η can be expressed as

$$\beta_\xi = a_1 + c\eta \;\; ; \;\; \beta_\eta = a_2 - c\xi \quad (25)$$

At point C (ξ = 1/2,η = 1/2),

$$\beta_s^C = \frac{1}{\sqrt{2}}\left\{\beta_\xi^C - \beta_\eta^C\right\}, \quad (26)$$

$$\beta_s^C = \frac{1}{\sqrt{2}}\left\{\left(\beta_\xi^A + \frac{1}{2}c\right) - \left(\beta_\eta^B - \frac{1}{2}c\right)\right\}, \quad (27)$$

$$c = \beta_\eta^{(B)} - \beta_\xi^{(A)} + \beta_\xi^{(C)} - \beta_\eta^{(C)}, \quad (28)$$

where

$$\beta_\xi^A = \frac{1}{2}\left(\beta_{\xi_1} + \beta_{\xi_2}\right) \;\; ; \;\; \beta_\eta^B = \frac{1}{2}\left(\beta_{\eta_1} + \beta_{\eta_3}\right)$$

$$\beta_\xi^C = \frac{1}{2}\left(\beta_{\xi_2} + \beta_{\xi_3}\right) \;\; ; \;\; \beta_\eta^C = \frac{1}{2}\left(\beta_{\eta_2} + \beta_{\eta_3}\right)$$

$$\begin{Bmatrix}\gamma_\xi \\ \gamma_\eta\end{Bmatrix} = \begin{Bmatrix} w_{,\xi} + \beta_\xi \\ w_{,\eta} + \beta_\eta \end{Bmatrix} = \begin{Bmatrix} w_2 - w_1 + \beta_\xi \\ w_3 - w_1 + \beta_\eta \end{Bmatrix}$$

The shear strain in parametric space can be expressed as

$$\begin{Bmatrix}\gamma_\xi \\ \gamma_\eta\end{Bmatrix} = \left[B_{s_\xi}\right]\{u_n\}, \quad (29)$$

$$\begin{Bmatrix}\gamma_\xi \\ \gamma_\eta\end{Bmatrix} = \begin{Bmatrix} w_2 - w_1 + \frac{1}{2}\left\{(x_{21})(\beta_{x1} + \beta_{x2}) + (y_{21})(\beta_{y1} + \beta_{y2})\right\} + c\eta \\ w_3 - w_1 + \frac{1}{2}\left\{-(x_{13})(\beta_{x1} + \beta_{x3}) - (y_{13})(\beta_{y1} + \beta_{y3})\right\} - c\xi \end{Bmatrix}$$

,

(30)

$$\left[B_{s_\xi}\right] = \frac{1}{2}\begin{Bmatrix} -2 & 1-\eta & \eta & 2 & 1 & -\eta & 0 & \eta & 0 \\ -2 & \xi & 1-\xi & 0 & 0 & \xi & 2 & -\xi & 1 \end{Bmatrix}, \quad (31)$$

and the transverse shear strain field in Cartesian system can be expressed as

$$\begin{Bmatrix}\gamma_x \\ \gamma_y\end{Bmatrix} = [j]\begin{Bmatrix}\gamma_\xi \\ \gamma_\eta\end{Bmatrix}, \quad (32)$$

where $[j]$ is the inverse of the Jacobian matrix

$$\begin{Bmatrix}\gamma_x \\ \gamma_y\end{Bmatrix} = [B_s]\{u_n\} \;\; ; \;\; [B_s] = [j]\left[B_{s_\xi}\right]. \quad (33)$$

II. NUMERICAL RESULT

A circular plate problem with different mesh directions is used to study the convergence behavior of the MITC3 element. A simple supported (SS) and clamped circular plate of radius R = 5 are modelled with eight meshes of 6, 24, 54, 96, 216, 384, 600, and 2400 elements. The thickness variations are h = 0.1, h = 1, h = 2, and h=2.5; the modulus of elasticity is E= 10.92; the Poisson ratio is 0.3; and the uniform load is fz = 1. The mesh distributions vary with the right and left directions.

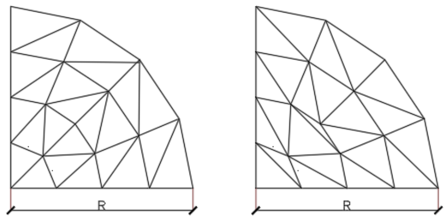

Fig. 4. Mesh direction of circular plate

Tables I and II present the central displacement of circular plates under simply supported and clamped boundary conditions. Fig. 5-12 shows the convergence of displacement.

TABLE I. CENTRAL DISPLACEMENT FOR THE SIMPLE SUPPORTED CIRCULAR PLATE UNDER UNIFORM LOADING

R/h	NELT	MITC3 (L)	MITC3 (R)	EXACT [4]
50	6	678.03	1502.7	39831
	24	35321	35305	
	54	37521	37785	
	96	38525	38612	
	216	39120	39141	
	384	39287	39294	
	600	39355	39358	
	2400	39435	39435	
5	6	8.6516	12.124	41.599
	24	39.251	39.415	
	54	40.359	40.451	
	96	40.75	40.808	
	216	41.031	41.06	
	384	41.13	41.147	
	600	41.176	41.188	
	2400	41.238	41.242	
2.5	6	2.1641	2.5506	5.87
	24	5.5482	5.5999	
	54	5.6991	5.7276	
	96	5.7535	5.7716	
	216	5.7931	5.8024	
	384	5.8073	5.8129	
	600	5.8139	5.8178	
	2400	5.823	5.8241	
2	6	1.4058	1.6266	3.262
	24	3.0849	3.1233	
	54	3.1682	3.1893	
	96	3.1985	3.2118	
	216	3.2207	3.2275	
	384	3.2286	3.2328	
	600	3.2324	3.2352	
	2400	3.2375	3.2383	

TABLE II. CENTRAL DISPLACEMENT FOR THE CLAMPED CIRCULAR PLATE UNDER UNIFORM LOADING

R/h	NELT	MITC3 (L)	MITC3 (R)	EXACT [4]
50	6	467.86	371.55	9784
	24	4475.5	4841.9	
	54	7793.9	7894.5	
	96	8913.3	8934.7	
	216	9471.1	9478.7	
	384	9598.9	9604.2	
	600	9645	9648.9	
	2400	9693.5	9694.6	
5	6	6.5774	6.6123	11.551
	24	10.148	10.299	
	54	10.883	10.968	
	96	11.139	11.193	
	216	11.321	11.348	
	384	11.385	11.401	
	600	11.415	11.426	
	2400	11.455	11.458	
2.5	6	1.4371	1.5327	2.114
	24	1.9099	1.9609	
	54	2.0128	2.0406	
	96	2.0502	2.0677	
	216	2.0776	2.0865	
	384	2.0875	2.0929	
	600	2.0921	2.0958	
	2400	2.0984	2.0995	
2	6	0.95439	1.0352	1.339
	24	1.2218	1.26	
	54	1.3022	1.3013	
	96	1.2806	1.3152	
	216	1.3022	1.3248	
	384	1.3239	1.328	
	600	1.3266	1.3294	
	2400	1.3303	1.3312	

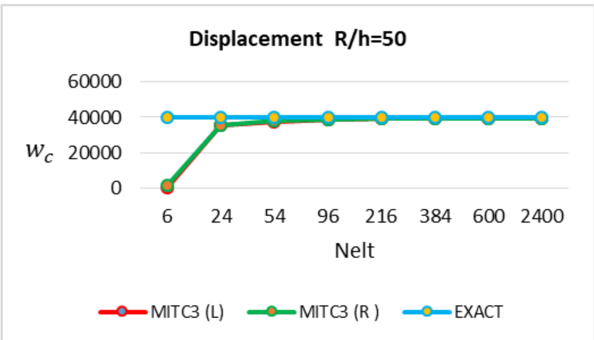

Fig. 5. Convergence of wc for simply supported problem with *R/h* = 50

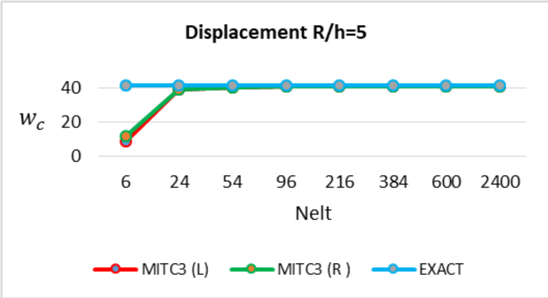

Fig. 6. Convergence of wc for simply supported problem with *R/h* = 5

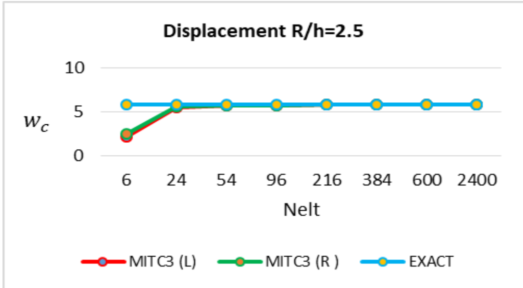

Fig. 7. Convergence of wc for simply supported problem with *R/h* = 2.5

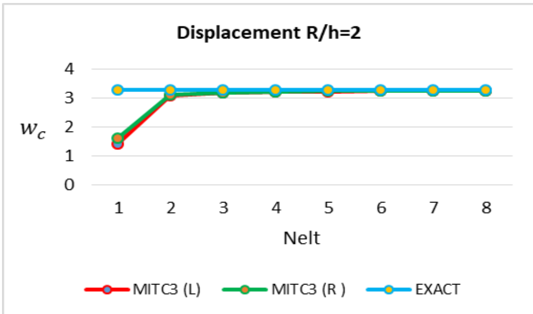

Fig. 8. Convergence of wc for simply supported problem with *R/h* = 2

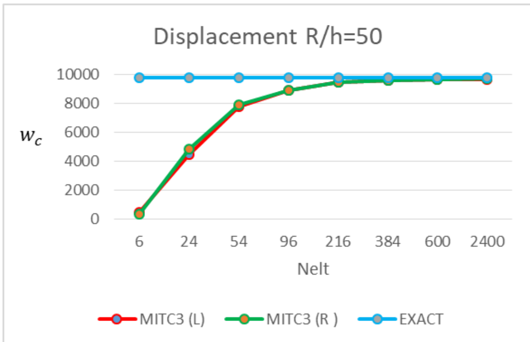

Fig. 9. Convergence of wc for clamped problem with *R/h* =50

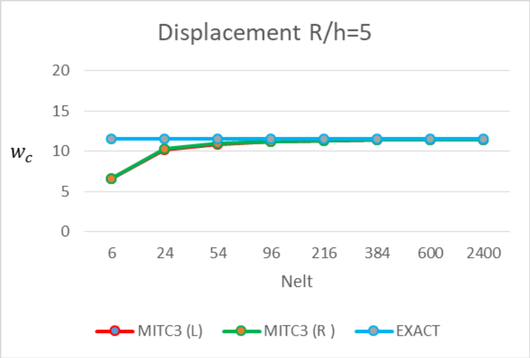

Fig. 10. Convergence of wc for clamped problem with *R/h* = 5

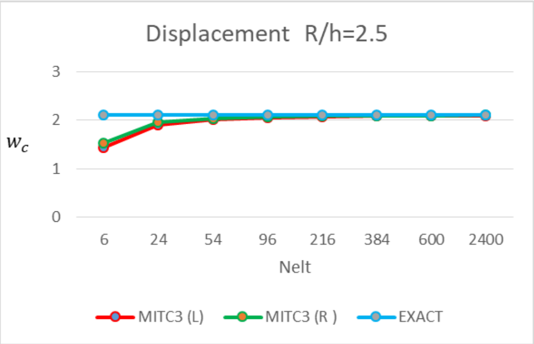

Fig. 11. Convergence of wc for clamped problem with *R/h* = 2.5

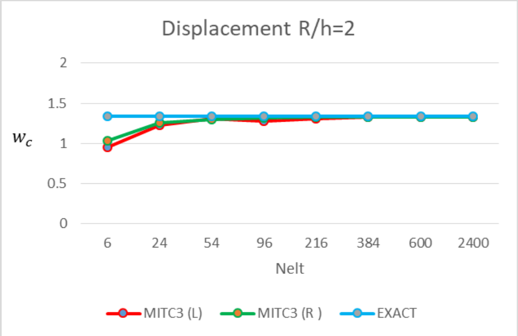

Fig. 12. Convergence of wc for clamped problem with *R/h* = 2

III. CONCLUSION

The element shows good convergence toward the exact solution. The right direction of the mesh provides faster convergence than the left direction and is clearly visible on thin plates. The clamped circular plate requires more meshes to reach convergence than the simple supported plate.

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