Comparison of Isogeometric Method and Finite Element Analysis on Beams Over Elastic Foundations Based on UI Approach

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***Abstract*—The UI-beam theory is a new thesis within the finite element method (FEM) and a modification of the Timoshenko beam. Isogeometric method is a development of FEM itself that uses B-spline function to replace the shape function in FEM. The capability of the isogeometric method to produce a perfect geometry curvature using only a few elements is its advantage. This study focuses on a single variable beam on the elastic foundation case which, later on, presents a comparison between FEM and the isogeometric method to obtain an exact solution.**

***Keywords*—UI-Beam, Isogeometric, B-Splines, Beam on Elastic Foundation, Single Variable**

I. Introduction

Achieving a nearly perfect exact solution and cross-section geometry is one of the major challenges in mechanical engineering. A beam is a 1D structure characterized by numerous obscure problems. A problem arises in calculating the beam above the placement of the elastic foundation where soil is assumed to be a spring along the beam span. In this case, the calculation would be inevitable from any impacts.

In this regard, finite element method (FEM) is one of the most advanced and precise ways to obtain the exact solution. FEM adequately provides an exact solution, which is the method to divide a beam into several elements. The only drawback of this method is that it requires many elements to achieve the same shape when faced with a curved section. Nevertheless, the problem has been settled by the isogeometric method. The B-spline in the method aims to substitute the shape function on FEM and also provides a perfect curve although using only a few elements. Thus, this paper presents a comparison of the results of using FEM, isogeometric method, and exact solution.

II. Methods

A. Formula Derivation of UI Beam

As shown by Katili [3] in the Timoshenko beam theory, the beam is affected not only by bending but also by shear.

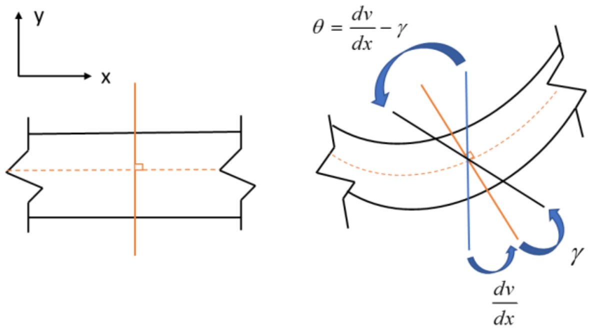

Figure 1. Timoshenko Beam Theory

Therefore, the shear deformation is not zero. Timoshenko found the relation between curvature, shear deformation, moment, and shear force as expressed by the following equations:

$$\chi = -\frac{\mathrm{d}\theta}{\mathrm{d}x} \tag{1}$$

$$\gamma = \frac{\mathrm{d}v}{\mathrm{d}x} - \theta \tag{2}$$

$$M = EI\chi, \tag{3}$$

$$T = kGA\gamma. \tag{4}$$

Furthermore, moment, shear force, and force relation are shown by the following equation:

$$\frac{\mathrm{d}M}{\mathrm{d}x} = T \tag{5}$$

$$\frac{\mathrm{d}T}{\mathrm{d}x} = f \cdot \quad (6)$$

Two shear force equations appear when one substitutes equation [3] to [5] and equation [2] to [4].

$$-EI\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + \kappa GA\left(\frac{\mathrm{d}v}{\mathrm{d}x} - \theta\right) = 0 \quad (7)$$

Using the first shear force equation in [7], if we substitute it to [6], then it becomes

$$-EI\frac{\mathrm{d}^3\theta}{\mathrm{d}x^3} = f \cdot \quad (8)$$

To obtain the first derivative of displacement, we operate [7] and present it as follows:

$$\begin{aligned}
&EI\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + \kappa GA\left(\frac{\mathrm{d}v}{\mathrm{d}x} - \theta\right) = 0\\
&EI\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} = -\kappa GA\left(\frac{\mathrm{d}v}{\mathrm{d}x} - \theta\right)\\
&EI\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} - \kappa GA\theta = -\kappa GA\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)\\
&-\frac{EI}{\kappa GA}\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + \theta = \frac{\mathrm{d}v}{\mathrm{d}x}
\end{aligned} \quad (9)$$

Then, we integrate [9].

$$v = \int_0^x \theta \mathrm{d}x - \frac{EI}{\kappa GA}\frac{\mathrm{d}\theta}{\mathrm{d}x} + c \cdot \quad (10)$$

In the Timoshenko beam theory, deformation is caused by bending and shear deformation.

$$v = v_b + v_s, \quad (11)$$

$$v_b = \int_0^x \theta \mathrm{d}x + c, \quad (12)$$

$$v_s = -\frac{EI}{\kappa GA}\frac{\mathrm{d}\theta}{\mathrm{d}x} + c \cdot \quad (13)$$

We operate the derivation of [11] as follows:

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}v_b}{\mathrm{d}x} + \frac{\mathrm{d}v_s}{\mathrm{d}x}, \quad (14)$$

$$\frac{\mathrm{d}v_b}{\mathrm{d}x} = \theta, \quad (15)$$

$$\frac{\mathrm{d}v_s}{\mathrm{d}x} = -\frac{EI}{\kappa GA}\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} = -\frac{EI}{\kappa GA}\frac{\mathrm{d}^3 v_b}{\mathrm{d}x^3} \cdot \quad (16)$$

We substitute [15] to [1].

$$\chi = -\frac{\mathrm{d}^2 v_b}{\mathrm{d}x^2} \quad (17)$$

Equation [1] is [14] minus [15].

$$\gamma = \frac{\mathrm{d}v_s}{\mathrm{d}x} = -\frac{EI}{\kappa GA}\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} = -\frac{EI}{\kappa GA}\frac{\mathrm{d}^3 v_b}{\mathrm{d}x^3} \cdot \quad (18)$$

We integrate [16].

$$v_s = -\frac{EI}{kGA}\frac{\mathrm{d}^2 v_b}{\mathrm{d}x^2} \quad (19)$$

Then, [19] is substituted to [11].

$$v = vb - \frac{EI}{kGA}\frac{\mathrm{d}^2 v_b}{\mathrm{d}x^2} \cdot \quad (20)$$

Furthermore, we substitute [16] to [8].

$$-EI\frac{\mathrm{d}^4 v_b}{\mathrm{d}x^4} = f \quad (21)$$

UI-beam has six degrees of freedom (DOFs), which are v, θ, and χ. Figure 2 illustrates the DOFs.

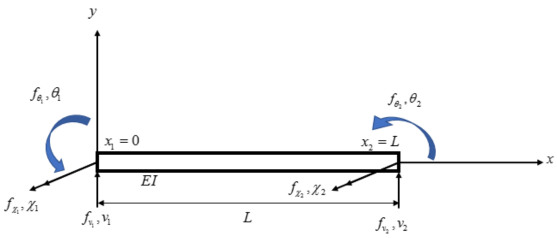

Figure 2. Degrees of freedom in UI-beam nodes

As a result of the six DOFs, the UI-beam uses the fifth degree of polynomial to obtain the shape function. The polynomial function appears as

$$\begin{aligned}
&v_b = \langle P \rangle \{a_n\} = \langle 1 \quad x \quad x^2 \quad x^3 \quad x^4 \quad x^5 \rangle \{a_n\};\\
&\{a_n\} = \langle a_n \rangle^T = \langle a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \rangle
\end{aligned} \cdot \quad (22)$$

The DOFs are v, θ, and χ. The first node is at zero, and the second is at the tip of the beam, and the distance is L.

$$\{u_n\} = [P_n]\{a_n\}$$

$$\begin{Bmatrix} v_{b1} \\ \theta_1 = \frac{dv_b}{dx}(x=0) \\ \chi_1 = -\frac{d^2v_b}{dx^2}(x=0) \\ v_{b2} \\ \theta_2 = \frac{dv_b}{dx}(x=L) \\ \chi_2 = -\frac{d^2v_b}{dx^2}(x=L) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 1 & L & L^2 & L^3 & L^4 & L^5 \\ 0 & 1 & 2L & 3L^2 & 4L^3 & 5L^4 \\ 0 & 0 & -2 & -6L & -12L^2 & -20L^3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

$$\{a_n\} = [P_n]^{-1}\{u_n\}$$

(23)

We substitute [23] to [22].

$$v_b = \langle P \rangle \{a_n\} = \langle P \rangle [P_n]^{-1}\{u_n\} = \langle N_{v_b} \rangle \{u_n\}. \quad (24)$$

From [24], we obtain the shape function and shape function graphic.

$$\begin{aligned} N_{v_{b1}} &= \frac{(L-x)^3(L^2+3Lx+6x^2)}{L^5} \\ N_{\theta_1} &= \frac{x(L-x)^3(L+3x)}{L^4} \\ N_{\chi_1} &= -\frac{x^2(L-x)^3}{2L^3} \\ N_{v_{b2}} &= \frac{x^3(10L^2-15Lx+6x^2)}{L^5} \\ N_{\theta_2} &= -\frac{x^3(4L^2-7Lx+3x^2)}{L^4} \\ N_{\chi_2} &= -\frac{x^3(L-x)^2}{2L^3} \end{aligned} . \quad (25)$$

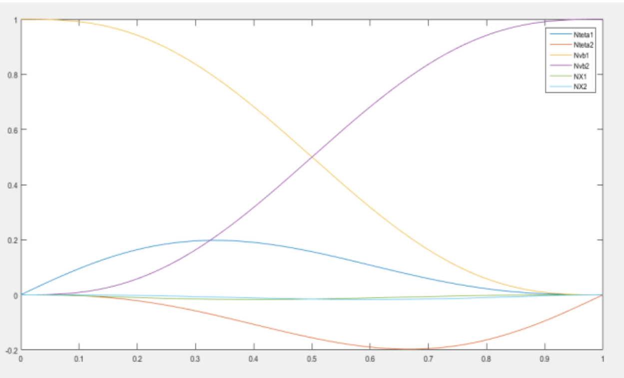

Figure 3. Shape function graphic

Thereafter, a strain matrix is calculated. Bending strain matrix is the curvature shown in [17].

$$\chi = -\frac{d^2v_b}{dx^2} = \langle B_b \rangle \{u_n\}$$

$$\langle B_b \rangle = -\left\langle \frac{d^2N_{v_{b_1}}}{dx^2} \quad \frac{d^2N_{\theta_1}}{dx^2} \quad \frac{d^2N_{\chi_1}}{dx^2} \quad \frac{d^2N_{v_{b_2}}}{dx^2} \quad \frac{d^2N_{\theta_2}}{dx^2} \quad \frac{d^2N_{\chi_2}}{dx^2} \right\rangle$$

(26)

Strain matrix consequence of shear deformation using [18].

$$\gamma = -\frac{EI}{kGA}\frac{d^3v_b}{dx^3} = \langle B_s \rangle \{u_n\}$$

$$\phi = \frac{EI}{kGA}\frac{12}{L^2}$$

$$\langle B_s \rangle = -\frac{\phi L^2}{12}\left\langle \frac{d^3N_{v_{b_1}}}{dx^3} \quad \frac{d^3N_{\theta_1}}{dx^3} \quad \frac{d^3N_{\chi_1}}{dx^3} \quad \frac{d^3N_{v_{b_2}}}{dx^3} \quad \frac{d^3N_{\theta_2}}{dx^3} \quad \frac{d^3N_{\chi_2}}{dx^3} \right\rangle$$

(27)

As the shape function has been obtained, the strain matrix should be discretized to transform it into the stiffness matrix. The discretization operation is

$$\begin{aligned} [k_b] &= EI\int_0^L \{B_b\}\langle B_b \rangle dx \\ [k_s] &= \int_0^L \{B_s\} kGA \langle B_s \rangle dx \end{aligned} . \quad (28)$$

The stiffness matrix of bending moment and shear deformation is

$$[k_b] = \frac{EI}{70L^3}\begin{bmatrix} 1200 & 600L & -30L^2 & -1200 & 600L & 30L^2 \\ & 384L^2 & -22L^3 & -600L & 216L^2 & 8L^3 \\ & & 6L^4 & 30L^2 & -8L^3 & L^4 \\ & & & 1200 & -600L & -30L^2 \\ & sym & & & 384L^2 & 22L^3 \\ & & & & & 6L^4 \end{bmatrix}$$

$$[k_s] = \frac{\phi EI}{4L^3}\begin{bmatrix} 240 & 120L & -20L^2 & -240 & 120L & 20L^2 \\ & 64L^2 & -12L^3 & -120L & 56L^2 & 8L^3 \\ & & 3L^4 & 20L^2 & -8L^3 & -L^4 \\ & & & 240 & -120L & -20L^2 \\ & sym & & & 64L^2 & 12L^3 \\ & & & & & 3L^4 \end{bmatrix}$$

. (29)

The stiffness matrix is the sum of the bending and shear stiffness matrices.

$$[k] = [k_b] + [k_s]$$

$$[k] = \frac{EI}{140L^3}\begin{bmatrix} 1200(2+7\phi) & 600L(2+7\phi) & -20L^2(3+35\phi) & -1200(2+7\phi) & 600L(2+7\phi) & 20L^2(3+35\phi) \\ & 64L^2(12+35\phi) & -4L^3(11+105\phi) & -600L(2+7\phi) & 8L^2(54+245\phi) & 8L^3(2+35\phi) \\ & & 3L^4(4+35\phi) & 20L^2(3+35\phi) & -8L^3(2+35\phi) & L^4(2-35\phi) \\ & & & 1200(2+7\phi) & -600L(2+7\phi) & -20L^2(3+35\phi) \\ & & sym & & 64L^2(12+35\phi) & 4L^3(11+105\phi) \\ & & & & & 3L^4(4+35\phi) \end{bmatrix}$$

. (30)

The external force is the integral of distributed loads and deformation along the element span. The deformation consists of bending and shear deformation. Thus, the external force is

$$\prod_{ext} = \int_0^L f_0 v(x)\mathrm{d}x = f_0 \int_0^L (v_b(x) + v_s(x))\mathrm{d}x = f_0 \int_0^L (v_b(x) - \frac{L^2\phi}{12}\frac{\mathrm{d}^2 v_b}{\mathrm{d}^2 x})\mathrm{d}x = \langle u_n \rangle \{ f_n \}$$

. (31)

B. Modification of UI Beam

The force consists of internal and external forces. In the beam on the elastic foundation case, internal force consists of bending, shear, and spring forces. The spring force appears to be

$$\prod_{sp} = \frac{1}{2}\int_0^L sv^2 \mathrm{d}x = \frac{1}{2}\langle u_n \rangle \left[k_{sp} \right] \{ u_n \} \quad (32)$$

The stiffness matrix is the integral along the span from a transformation of the deformation matrix multiplied with the deformation matrix. This deformation matrix consists of bending and shear deformation; the former has been described in [24] and the latter in [19]. Thus, the stiffness matrix can be stated in

$$\left[k_{sp} \right] = s\int_0^L \left\{ \langle N \rangle - \frac{EI}{kGA}\frac{\mathrm{d}^2 \langle N \rangle}{\mathrm{d}x^2} \right\} \left\langle \langle N \rangle - \frac{EI}{kGA}\frac{\mathrm{d}^2 \langle N \rangle}{\mathrm{d}x^2} \right\rangle \mathrm{d}x$$

. (33)

To simplify the preceding equation, we created the strain matrix where value is equal to displacement.

$$\begin{aligned} \langle B_{sp} \rangle &= \langle N \rangle - \frac{EI}{kGA}\frac{\mathrm{d}^2 \langle N \rangle}{\mathrm{d}x^2} \\ \langle B_{sp} \rangle &= \langle N \rangle - \frac{\phi L^2}{12}\frac{\mathrm{d}^2 \langle N \rangle}{\mathrm{d}x^2} \\ \left[k_{sp} \right] &= s\int_0^L \{ B_{sp} \} \langle B_{sp} \rangle \mathrm{d}x \end{aligned} \quad (34)$$

Then, the result of the spring stiffness matrix is

$$\left[k_{sp}\right] = \frac{sL^5}{462}\begin{bmatrix} \frac{(55\phi^2+110\phi+181)}{L^4} & \frac{(275\phi^2+550\phi+311)}{10L^3} & -\frac{(165\phi^2+110\phi+281)}{120L^2} & -\frac{5(11\phi^2+22\phi-10)}{L^4} & \frac{(275\phi^2+165\phi-151)}{10L^3} & \frac{(165\phi^2+110\phi-181)}{120L^2} \\ & \frac{8(33\phi^2+33\phi+13)}{15L^2} & -\frac{(121\phi^2+154\phi+69)}{120L} & -\frac{(275\phi^2+165\phi-151)}{10L^3} & -\frac{(-297\phi^2+33\phi+133)}{30L^2} & -\frac{(-11\phi^2+11\phi+13)}{30L} \\ & & \frac{(99\phi^2+44\phi+18)}{360} & \frac{(165\phi^2+110\phi-181)}{120L^2} & \frac{(-11\phi^2+11\phi+13)}{30L} & \frac{(33\phi^2+44\phi+30)}{720} \\ & & & \frac{(55\phi^2+110\phi+181)}{L^4} & -\frac{(275\phi^2+550\phi+311)}{10L^3} & -\frac{(165\phi^2+110\phi+281)}{120L^2} \\ & & sym & & \frac{8(33\phi^2+33\phi+13)}{15L^2} & \frac{(121\phi^2+154\phi+69)}{120L} \\ & & & & & \frac{(99\phi^2+44\phi+18)}{360} \end{bmatrix}$$

. (35)

Then, the new stiffness matrix is the sum of the bending stiffness matrix, shear stiffness matrix, and spring stiffness matrix.

$$\left[k \right] = \left[k_b \right] + \left[k_s \right] + \left\lfloor k_{sp} \right\rfloor \quad (36)$$

C. Isogeometric Method

As in the beam case, the beam on the elastic foundation can be solved by using B-splines with the formula shown by Aristion [1] as

for $p = 0$

$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \le \xi < \xi_{i+1} \\ 0 & \text{for the other} \end{cases}$$

for $p \ge 1$

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi)$$

(37)

III. Results and Discussion

For the numerical test, this study applies the FEM on the DSB and UI-beam, isogeometric, and exact solution. The problem is a beam on the elastic foundation that has 10 m length (L), modulus of elasticity (E) of $2x10^9 N/m^2$, inertia moment (I) of $4.5x10^{-4} m^4$, and shear correction function (κ) of $5/6$ because the cross-section is a rectangle with shear modulus (G) of $8x10^8 N/m^2$, cross section area (A) of $6x10^{-2} m^2$, spring constant (s) of $5.76 \ x \ 10^4 N/m^2$, uniform load of (f_0)$1000 \ N/m$, and beam in rigid body motion condition.

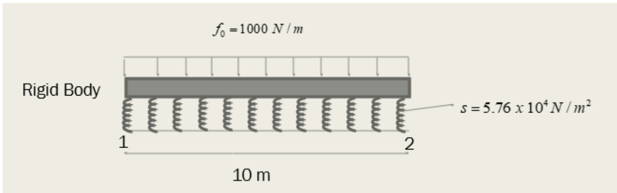

Figure 4. Example of a rigid body case

The boundary condition for this case is curvature (χ) and rotation (θ) at zero in both nodes. The results are presented in Table I.

TABLE I. Results of a rigid body case

Metode	Displacement function	v_1	v_2
DSB	5/288	0.0174	0.0174
UI-Beam	5/288	0.0174	0.0174
Iso (p=3)	$\frac{x(x/10-1)^2}{192} - \frac{x^2(x/10-1)}{1920} - \frac{5(x/10-1)^3}{288} + \frac{x^3}{57600}$	0.0174	0.0174
Iso (p=5)	$\frac{3x^2(x/10-1)}{1280000} + \frac{5x(x/10-1)^4}{576} - \frac{x^4(x/10-1)}{115200} - \frac{3x^2(x/50-1/5)}{256000} - \frac{5(x/10-1)^5}{288} - \frac{x^2(x/10-1)^3}{576} + \frac{x^3(x/10-1)^2}{5760} + \frac{x^5}{5760000}$	0.0174	0.0174
exact	$\frac{7605961479918485*\cos((3203463395364767*x)}{9007199254740992} \text{ x } \left(\frac{\cosh((3203463395364767x)/9007199254740992)}{292066698292584054104840418518630 4} \right) + 5/288$	0.0174	0.0174

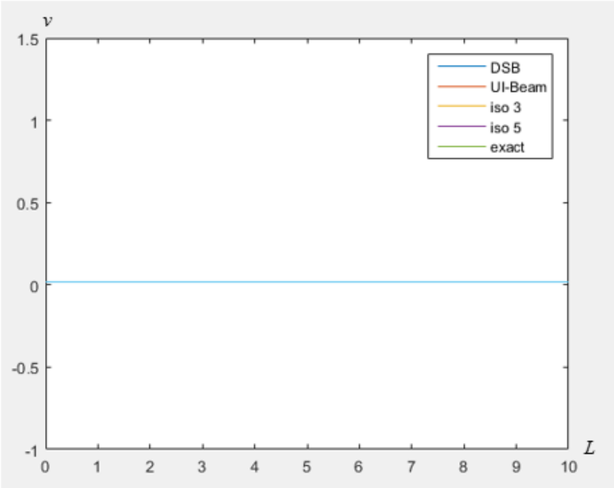

Figure 5. Displacement function of a rigid body case

From Table [I] and Fig. [5], we can see that the DSB, UI-beam, and isogeometric methods have the same displacement as the exact solution from Hetenyi [2] although the displacement function is different.

The second case consists of the same details but without rigid body motion.

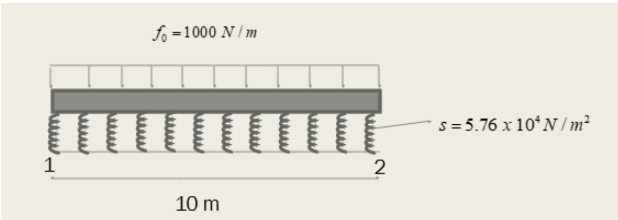

Figure 6. An example of a non-rigid body case

This problem has no curvature at both nodes. Thus, the results are

TABLE II. Results of a non-rigid body case

Metode	Displacement function	v_1	θ_1	v_2	θ_2
DSB	$-(675x^3)/18063037385457585 3568$ $-(7804040649098230 7x^2)/57801719633464273 14176$ $+(3902020324559915 35x)/28900859816732136 57088$ $+5/288$	0.0174	0.0001	0.0174	0.0001
UI-Beam	$(19245230883x^5)/1801439850948198400000$ $-(769857858 71x^4)/288230376151711744000$ $+(63989718379553x^3)/3602879701896396800000 0$ $+(2078616218517x^2)/2882303761517117440000 0$ $-(5776448930494 1x)/2882303761517117440000$ $+625590102460779/3602879701896396 8$	0.0174	0	0.0174	0
Iso (p=3)	$\frac{x(x/10-1)^2}{192}-\frac{x^2(x/10-1)}{1920}-\frac{5(x/10-1)^3}{288}+\frac{x^3}{57600}$	0.0174	0	0.0174	0
Iso (p=5)	$\frac{3x^2(x/10-1)}{1280000}+\frac{5x(x/10-1)^4}{576}-\frac{x^4(x/10-1)}{115200}-\frac{3x^2(x/50-1/5)}{256000}-\frac{5(x/10-1)^5}{288}-\frac{x^2(x/10-1)^3}{576}+\frac{x^3(x/10-1)^2}{5760}+\frac{x^5}{5760000}$	0.0174	0	0.0174	0
exact	$\frac{7605961479918485\cos((3203463395364767x)}{9007199254740992}$ $\times\left(\frac{\cosh((3203463395364767x)/9007199254740992)}{29206669829258405410484041851863 04}\right)$ $+5/288$	0.0174	0	0.0174	0

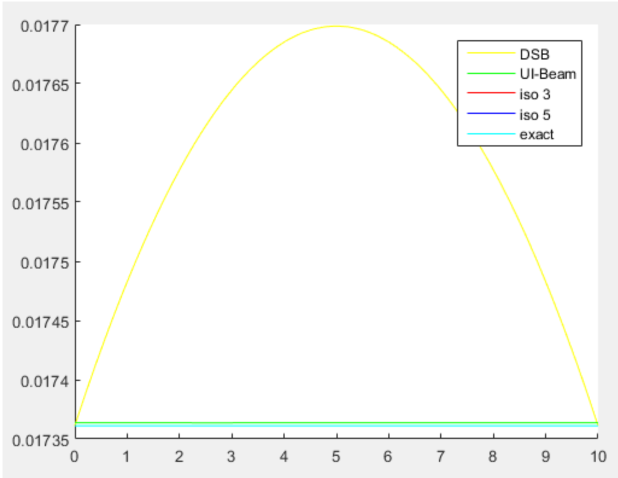

Figure 7. Displacement function of a non-rigid body case

From Table [II] and Fig. [7] of the second case, we can conclude that the isogeometric method provides the same displacement as the exact solution from Hetenyi [2], although the displacement function is different. On the other hand, the DSB seems to result from different displacements of the beam. UI-beam appears to provide the exact displacement but not the same. If we only plot the UI-beam displacement, the graphic is as shown in Fig. 8.

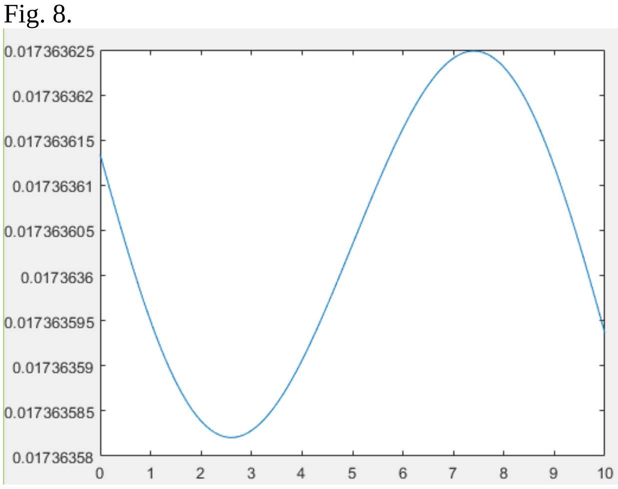
Figure 8. Displacement function of UI-beam for a non-rigid body case

The error from the UI-Beam is minimal, which makes it appear the same as the exact displacement. However, DSB and UI-beam provide the exact displacement on the nodes.

IV. Conclusions

This paper presents two conclusions. First, in the case of a rigid body of the beam on elastic foundation, the DSB, UI-beam, and isogeometric methods provide the displacement similar to the exact solution. Second, in the case of a non-rigid body, the resulting displacement just about the exact solution only occurs in the isogeometric method. The DSB, however, does not provide the exact results along the span. This method, in this case, provides the exact solution only on the node.

The UI-beam has a small error in the displacement function.

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